displacements need not. Subject only to geometric constraints (usually) an analyst is free to pick anything he wants for virtual displacements; and with this freedom, it is very easy to make $(\partial T/\partial q_i)\delta q_i$ equal to zero at any time simply by selecting $\delta q_i = 0$ at that time.

Professor Bailey states that "the expression which is always an extremum is the time integral of the virtual work of the system." This statement is not precise. The fact that Professor Bailey's Eq. (2) indicates a certain quantity equal to zero is not the same as saying that the certain quantity is an extremum. In our paper we do not say that Eq. (1) is the expression which is stationary but rather the quantity I defined by Eq. (2) is stationary because $\delta I = 0$.

Hamilton's "Law of Varying Action" as presented in 1834 and 1835 is the basis of what has since developed into the Hamilton-Jacobi theory (p. 88 of our Ref. 4). This is quite different from the application alluded to by Professor Bailey, whose method, as described in his Ref. 4, is simply an incremental, weighted residual (in particular, Galerkin's method) approximate solution to the differential equations of motion, with trial functions which satisfy initial conditions. Professor Bailey seems to prefer an interpretation in terms of the principle of virtual work because there need be no reference to differential equations. While we agree that virtual work provides a powerful starting point for problems in mechanics, it must be recognized that the differential equation formulation is completely equivalent when necessary continuity conditions are satisfied. Finally, it should be noted that Professor Bailey's approximate solution technique recovers the exact answer to his last example because his family of trial functions happens to include the exact solution. Further comment on his procedure will be more appropriate at another time in another place.

There are two errors in our paper which we take this opportunity to correct. In Eq. (28), the Q_i should be preceded with a minus sign. Then in Eq. (29), the $-Q_i$ should be $+2Q_i$.

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Further Comment on "Mathematical Modeling of Spinning Elastic Bodies for Modal Analysis"

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THE equation of flexural motion of a spinning radial beam is derived in Ref. 1 as an illustration of "partial linearization" advocated by Professor Likins and his associates in describing the structural mode of motion of a flexible spacecraft. In commenting on Ref. 1, Dr. Vigneron[2] indicates the same equation may be obtained in an alternative approach. The two approaches have fundamental disagreements, i.e., the Vigneron derivation considers the change in the kinetic energy of rotation as a result of antenna deflection but neglects the extensional strain energy. The Likins derivation takes exactly the opposite point of view. The formal agreement of the resulting equation could only be considered a coincidence and a closer examination is called for.

For clarity, the Vigneron notation shall be followed. Bending shall be considered in one plane only, i.e., $\eta = 0$. The starting point for both derivations is the Hamilton's principle. The spin is held fixed thus excluding any possibility of the "tail-wags-dog" phenomenon. An Euler-Bernoulli beam is considered. Implicitly or explicitly, the squares of the rotation of the beam cross-section are considered important. Therefore, as it is customary in the "large deflection" of beams, the centroidal axis experiences the strain

$$\epsilon_{yy} = \frac{\partial v}{\partial y} + \frac{1}{2}\left[\frac{\partial \xi}{\partial y}\right]^2 = \frac{\partial \zeta}{\partial y} \tag{1}$$

This equation may be written as

$$\frac{\partial \zeta_0}{\partial y} + \frac{\partial \zeta_1}{\partial y} = \frac{\partial v_0}{\partial y} + \frac{\partial v_1}{\partial y} + \frac{1}{2}\left[\frac{\partial \xi}{\partial y}\right]^2 \tag{2}$$

where, following Likins, the subscript 0 denotes a quantity related to the "undeflected spinning state." The basic difference of the two derivations is that Vigneron considers the centroidal axis inextensible when the beam deflects, i.e.,

$$+\frac{\partial \zeta_1}{\partial y} = \frac{\partial v_1}{\partial y} + \frac{1}{2}\left[\frac{\partial \xi}{\partial y}\right]^2 \approx 0$$

while Likins implicitly assumes $\partial v_1/\partial y \sim 0$, thus implying the centroidal axis experiences additional extension but that elements of the beam will not come closer to the spin axis as a result of deflection. Subsequent derivations of Likins and Vigneron are consistent with their respective assumptions.

Although the ultimate resolution of the issue depends on experimental evidence, it should be observed that: 1) The Likins assumption is inconsistent when the axial equation of motion is considered. 2) The Vigneron derivation really does not depend on the suppression of extensional motion. By considering variations in ζ in the energy expressions given in Ref. 2, one would obtain the following set of *linear* equations, a) the equation for the extensional motion. b) the same flexural equation of motion with the addition of a term representing Coriolis coupling from the extensional motion. As a matter of fact, one may assess the accuracy of the Vigneron assumption based on these equations.

The fact that the Likins derivations happens to give the correct equation of flexural motion may be explained as follows: 1) As noted previously, except for a minor Coriolis coupling, the flexural motion is relatively independent of the extensional motion. 2) The Likins derivation may be interpreted in terms of a formulation in a rotating frame of reference. The kinetic energy of rotation may then be replaced by the centrifugal potential energy which is precisely the term Likins retains as "extensional strain energy."

A key element in the Vigneron formulation is the use of the extensional displacement ζ in place of the radial displacement v as a generalized coordinate. The nonlinear term $\frac{1}{2}/(\partial\xi/\partial y)^2$ in Eq. (1) is absorbed in ζ so that although the strain energy in terms of ξ and v are non-quadratic, it becomes quadratic in ξ and ζ and therefore leads to linear equations. Physically this means that although the deflection and the radial motion is coupled when "large deflection" is considered, the coupling between deflection and extensional motion is weaker and the choice of these as variables extends the range of linearity. This is, of course, very attractive and

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one would immediately ask whether this formulation may be used for similar large deflection problems such as buckling. The answer is that although for a certain class of problems this may indeed be advantageous, the general applicability is limited by the fact that if boundary conditions are imposed on the radial displacement

$$v(\ell) = \zeta(\ell) - \frac{1}{2}\int_0^\ell \left[\frac{\partial\xi}{\partial y}\right]^2 dy \tag{3}$$

then the boundary condition becomes nonlinear although the differential equations remain linear. This point also underscores the general necessity of considering simultaneously the flexural and axial degrees of freedom in large deflection problems. It should also be pointed out that the Vigneron formulation trades the advantage of kinematic orthogonality ($\dot\xi$ and $\dot v$ are orthogonal velocity componets, $\dot\xi$ and $\dot\zeta$ are not) for structural simplicity. Conceivably there may be problems for which such trades are not worthwhile.

This commentator is in agreement with Professor Likins that the "partial linearization" method will remain one of the most useful methods. Unfortunately the radial beam example given in Ref. 1 represents a somewhat incorrect application of the method. Dr. Vigneron's formulation is certainly a viable alternative approach which deserves further exploration.

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Reply to Bertrand T. Fang

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DR. FANG correctly points out that the method outlined in Ref. 2 does not depend on the suppression of the extensional displacement, ζ. The suppression is a convenient approximation, which may be introduced to shorten the derivation somewhat.

Dr. Fang further notes that the boundary condition on $v(\ell)$ is nonlinear (Eq. (3) of his Comment). This does not necessarily preclude the possibility of use of the formulation for buckling problems. As an example, one may consider the case where Ω and $\partial(\ )/\partial t$ are equal to zero, bending is in one plane only ($\eta=0$), and a constant axial load, P, is applied at $y=\ell$. The work done by the load (negative potential) is the proportional to $-Pv(\ell)$, or

$$-P\{\zeta(\ell) - ½\int_0^\ell \left(\frac{\partial\xi}{\partial y}\right)^2 dy - \zeta(0)\}$$

Application of the principle of virtual work, with the above and the correspondingly simplified potential of Eq. (6), Ref. 2, leads to

$$EA\,\zeta_y = 0$$

$$EI\,\xi_{yyyy} - P\,\xi_{yy} = 0,$$

together with appropriate boundary conditions. The latter equation may be recognized as that associated with the study of buckling of beam columns. More general extensions of the formulation along these lines are given in Ref. 2.

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Comment on "On the Issue of Resonance in an Unsteady Supersonic Cascade"

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VERDON and McCune[1] have reported that the iterative procedures used in their linear unsteady supersonic cascade analysis failed to converge for certain combinations of the cascade parameters. The range in which divergence occurred was given as

$$|\sigma + kMx_A| \le k(x_A^2 - \mu^2 y_A^2)^{½} \tag{1}$$

where $k=\omega M\mu^{-2}$, $\mu^2 = M^2 - 1$, M is the freestream Mach number, x_A and y_A are the cascade stagger and normal gap distances, respectively, and ω and σ are the frequency and interblade phase angle of the blade motion. (Note that only an interblade phase angle variation of 2π is of physical interest and hence σ can be restricted to the range $-\pi<\sigma\le\pi$.) Since the publication of Ref. 1, numerical procedures have been developed by the present author which provide results within the foregoing range, but not at its end points

$$\sigma + kMx_A = \pm k(x_A^2 - \mu^2 y_A^2)^{½} \tag{2}$$

At these points the numerical approach used for evaluating the sum of an infinite series kernel function, given by Eq. (24) of Ref. 1, fails. On this basis it was suspected that this infinite series might be divergent for such parametric combinations, indicating resonant operation; however, this conjecture could not be proved. Thus, the proof appearing in Ref. 2 on the divergence of the kernel function for parameter values satisfying Eq. (2) is a most welcome contribution. This work is particularly useful since it generalizes the earlier work of Samoilovich,[3] which was only recently brought to this author's attention by Kurosaka. Further, Samoilovich's result appears to have been achieved by a formal mathematical demonstration rather than by a rigorous proof.

Resonance appears to be the logical consequence of linear theories. The conditions given by Eq. (2), or more generally by Eq. (10) of Ref. 2, are formally identical to those obtained for a subsonic cascade. In addition, resonance has the same physical interpretations for both subsonic and supersonic flows, including the result that the blades cannot support un-

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